‘TOUCANS’

Links between turbulence and diffusion. Choice of the additional prognostic equation(s). Third order moments' inclusion.

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Papers with results used in the present lecture

- Redelsperger, Mahé & Carlotti, 2001. *Boundary Layer Meteorology*, **101**, pp. 375-408.
- Zilitinkevitch, Elperin, Kleeorin, Rogachevskii, Esau, Mauritsen & Miles, 2008. *Quart. J. Roy. Meteor. Soc.*, **134**, pp. 793-799.
- Sukoriansky, Galperin & Staroselsky, 2005. *Phys. Fluids*, **17**, 085107, pp. 1-28.
- Galperin, Sukoriansky & Anderson, 2007. *Atmos. Sci. Lett.*, **8**, pp. 65-69.
- Cheng, Canuto & Howard, 2002. *J. Atmos. Sci.*, **59**, pp. 1550-1565.
- Canuto, Cheng, & Howard, 2007. *Ocean Modelling*, **16**, pp. 28-46.

Content of the present talk

- About the acronym and the sequence of talks
- A crash course on QNSE
- Brief digression on ‘anisotropy’
- Some considerations for the link (& separation) between turbulent and diffusive computations
- Is there something like a ‘critical R_i’?
- How many additional prognostic equations for the turbulent scheme?
- General issues concerning ‘non locality’
- The special case of TOUCANS Third Order Moments (TOMs) => preliminary choices

The acronym (this morning's talk)

Third

TOM's inclusion

Order moments

Unified

Prognostic equations

Condensation

Accounting and

Link Turb. ⇔ Diff.

N-dependent

Solver (for turbulence and diffusion)

The acronym (this afternoon's talk)

Third

Order moments

Unified

Condensation

Accounting and

N-dependent

Solver **(for <u>turbulence</u> and diffusion)**

The acronym (to-morrow's talk)

Third

Order moments

Unified

Condensation

Accounting and

N-dependent

Solver **(for turbulence and diffusion)**

The Reynolds formalism and its closure problem (1/3)

- Let us start with a simple vertical advection equation

$$\partial \psi / \partial t = -w(\partial \psi / \partial z)$$

- Introducing the fluctuations of the variables via

$$\psi = \overline{\psi} + \psi' \qquad w = \overline{w} + w'$$

- We may then develop, average, slightly simplify and introduce density effects

$$\partial(\overline{\psi} + \psi') / \partial t = -(\vec{w} + w')[\partial(\overline{\psi} + \psi') / \partial z]$$

$$\partial \overline{\psi} / \partial t + \vec{w}(\partial \overline{\psi} / \partial z) = -\partial \psi' / \partial t - w'(\partial \overline{\psi} / \partial z) - \vec{w}(\partial \psi' / \partial z) - w'(\partial \psi' / \partial z)$$

$$d\overline{\psi} / dt = -\partial\overline{\psi'} / \partial t - \overline{w'}(\partial \overline{\psi} / \partial z) - \vec{w}(\partial\overline{\psi'} / \partial z) - \overline{w'(\partial \psi' / \partial z)}$$

$$d\overline{\psi} / dt = -\overline{w'(\partial \psi' / \partial z)} = -\partial(\overline{w'\psi'}) / \partial z + \overline{\psi'(\partial w' / \partial z)} \approx -\partial(\overline{w'\psi'}) / \partial z$$

$$\Leftrightarrow \quad \frac{d\overline{\psi}}{dt} = -\frac{1}{\rho}\frac{\partial(\overline{\rho w'\psi'})}{\partial z}$$

The Reynolds formalism and its closure problem (2/3)

- So the evolution of any conservative prognostic variable requires (in first approximation) the knowledge of the statistical correlation of its fluctuations with those of the vertical velocity (for 1D turbulence).
- One may try and diagnose those first order correlations (lower order closure) but one may also write prognostic equations for them.
- The problem is that these equations for the second order statistical moments will automatically generate third order moment terms, and so-on.
- Furthermore, at every step in this process the number of equations and of higher order moment terms dramatically increases. So we need a closure to stop this proliferation.

The Reynolds formalism and its closure problem (3/3)

- Most full turbulent systems relate the fourth order moments to the lower ones with stationarised equations.
- This closes the system but the outcome remains awfully complicated.
- There is now an avenue of research pushing the same step one level downwards => hope of some rather simplified specification on third order moments for the equations of the second order moments.
- The system can even be further simplified if one selects only two second order moments for scrutiny, i.e. the turbulent kinetic energy (**TKE**) and the potential equivalent (**TPE**). One may also consider the sum of them (total turbulent energy) **TTE=TKE+TPE**
- The issue is open whether or not something beyond TKE should be treated prognostically (while TKE should be if we want to escape the purely diagnostic closure).

The QNSE new alternative (first paper, Sukoriansky, Galperin and Staroselsky, 2005)

- ***QNSE*** (Quasi-Normal Scale Elimination) does not follow the path of Reynolds. It rather considers the turbulence problem from a spectral point of view, thus putting 'waves' in the heart of the mathematical derivation.
- The results are directly obtained in terms of horizontal length scales. It then requires a 'filtering stationarity equation' (***not used further***) to put them in terms of Richardson numbers (for RANS-type applications).
- This has three consequences:
  - Total separation of the stability dependency functions between momentum and heat (=> impossibility to 'isolate' the anisotropy part in the latter);
  - No 'critical $\boldsymbol{R_i}$' exists when going to very stable regimes;
  - QNSE does not apply to the unstable regimes (or at least to the strongly unstable ones).
- The ensuing 6 viewgraphs (courtesy of Boris Galperin) give a flavour of all this.

Prelude

- "Internal wave literature is a disordered mélange… there is an intrinsic disorder in that the various elements are usually not connected to each other in a systematic way. The most intrinsically difficult piece of a synthetic treatment is to account for nonlinearity and dissipation in the wave field" (Polzin, 2004)
- The QNSE model has been developed to bring some order to this mélange by systematically accounting for anisotropic turbulence and internal waves
- The model is capable of accurate prediction of various flow characteristics including turbulence spectra, surface fluxes, 3D velocity and temperature distributions, turbulence variables, etc.
- The model presents a viable alternative to Reynolds stress closures and is free of their limitations

Stable stratification. Basics of the theory I

We consider a fully three-dimensional turbulent flow field with imposed vertical, stabilizing temperature gradient. The flow is governed by

momentum
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} - \alpha g T \hat{e}_3 = \nu_0 \nabla^2 \mathbf{u} - \frac{\nabla P}{\rho} + \mathbf{f}_0$$

temperature
$$\frac{\partial T}{\partial t} + (\mathbf{u}\nabla)T + \frac{d\Theta}{dz} u_3 = \kappa_0 \nabla^2 T$$

continuity
$$\nabla \mathbf{u} = 0$$

equations in Boussinesq approximation

Central problem: treatment of nonlinearity. Try perturbative solution based on expansion in powers of Re?

It is strongly divergent!

Less severe problem: the system is coupled

Basics of the theory II

Spectral approach is most suitable to deal with both problems

General idea: Re is small for smallest scales of motion →

- Derive perturbative solution for these small scales
- Assuming that these modes have a quasi-normal statistics, we ensemble-average a small band of these modes thus eliminating them from the primitive equations. This scale elimination produces small corrections to viscosity and heat diffusivity making them flow-dependent
- Viscosity increases – but effective Re remains O(1)
- Repeat this procedure for next band of smallest scales
- QNSE theory – recursive quasi-normal scale elimination

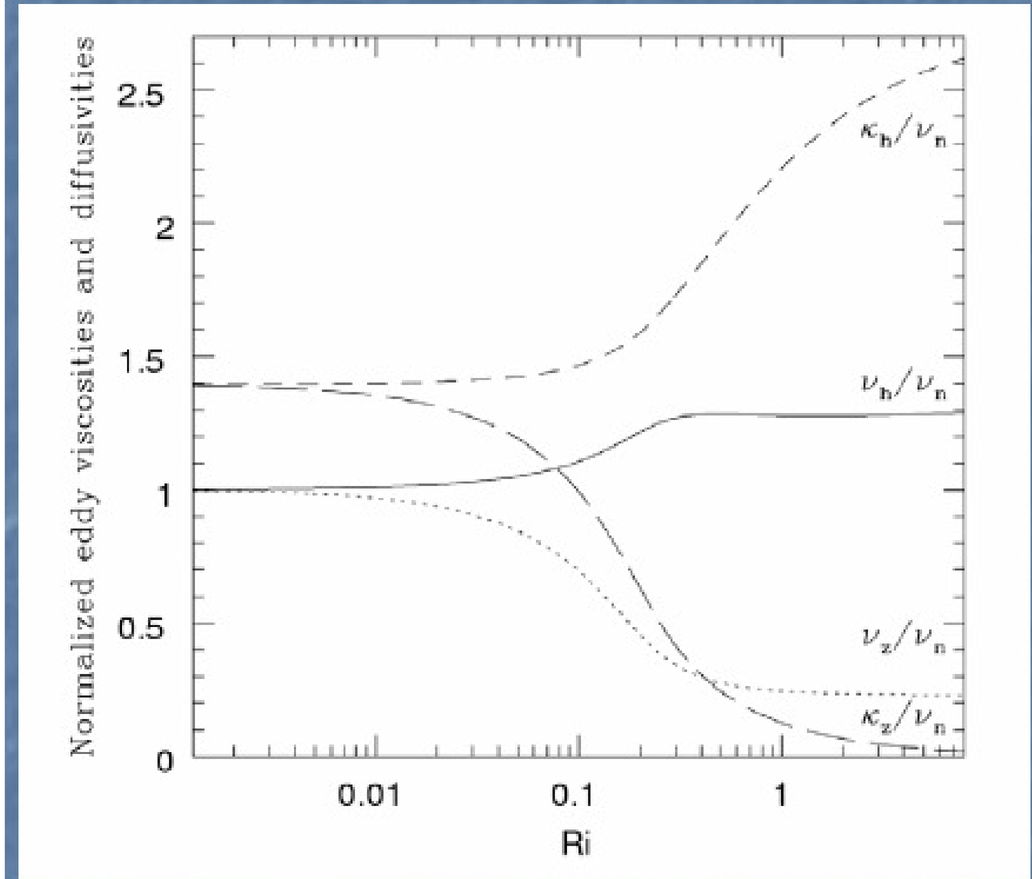
Basics of the theory III

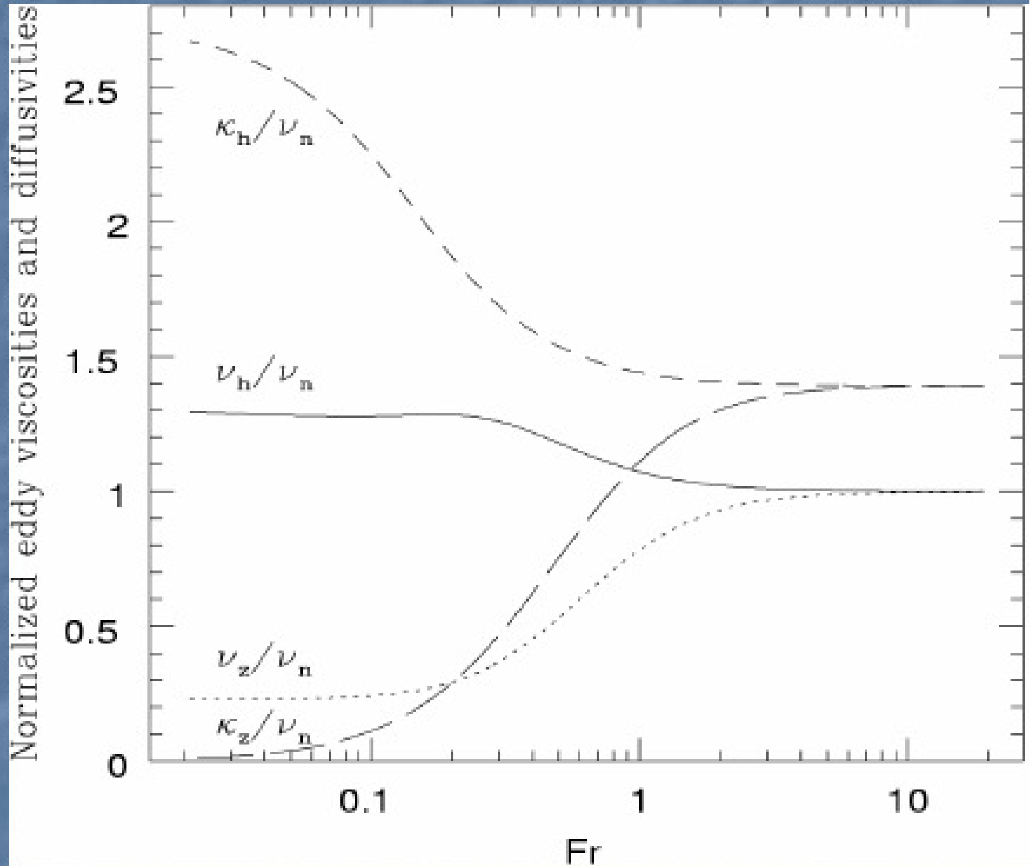
Partial scale elimination yields a subgrid-scale model for LES; complete scale elimination yields eddy viscosities and eddy diffusivities for RANS (Reynolds-average Navier-Stokes) models.

In either case, we don't separate turbulence and waves and treat them as one entity

RANS modeling

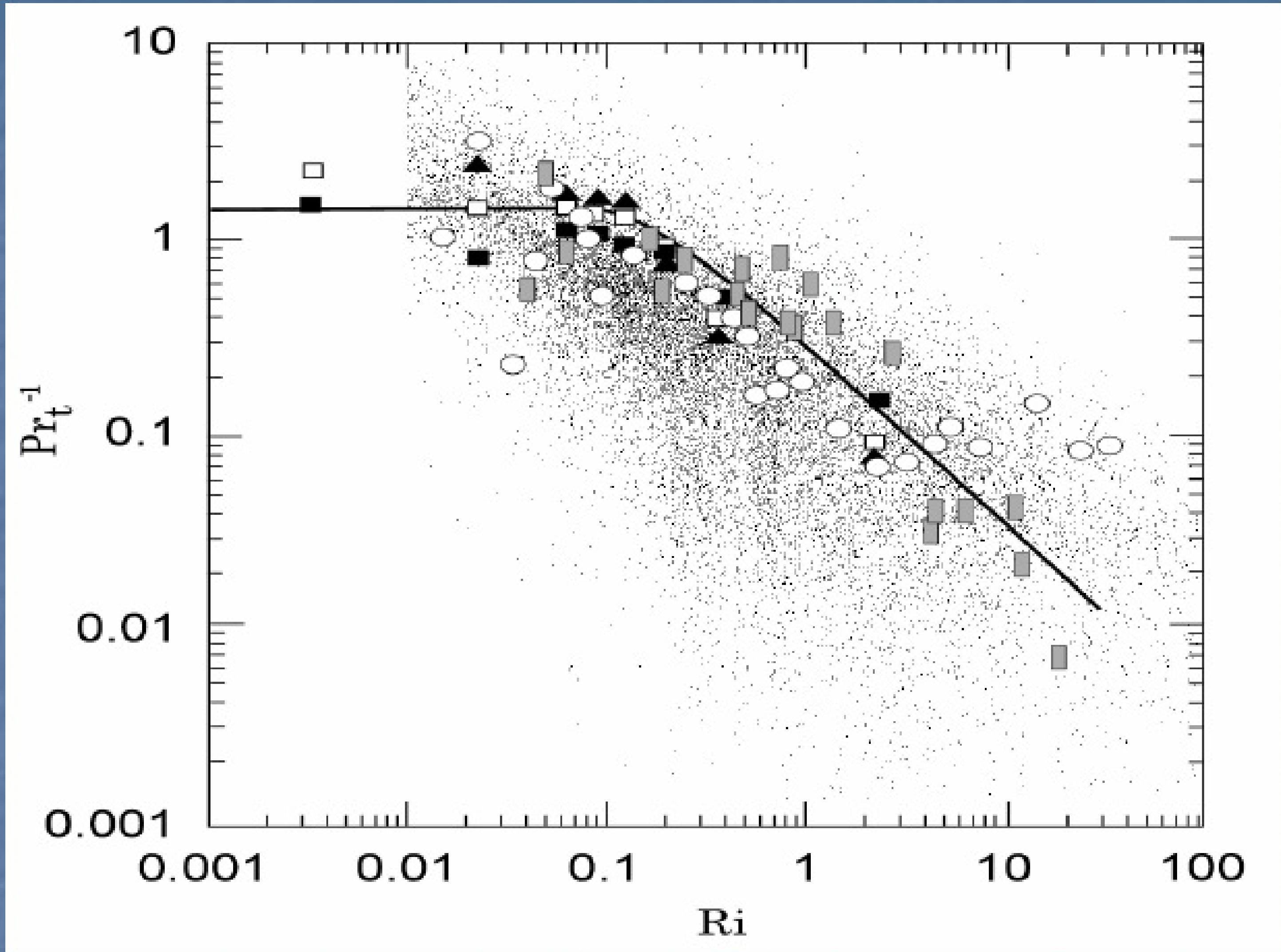
Eliminate all fluctuating scales; recast turbulent exchange coefficients as functions of either **the gradient Richardson number, $Ri = N^2/ S^2$, or the Froude number, $Fr = \varepsilon / NK$**





Vertical eddy viscosity and diffusivity decrease with Ri, while their horizontal counterparts increase. Due to the momentum mixing by internal waves, vertical viscosity (1) decreases slower than vertical diffusivity and (2) remains finite even at very large Ri

Comparison with data: Pr_t as a function of Ri

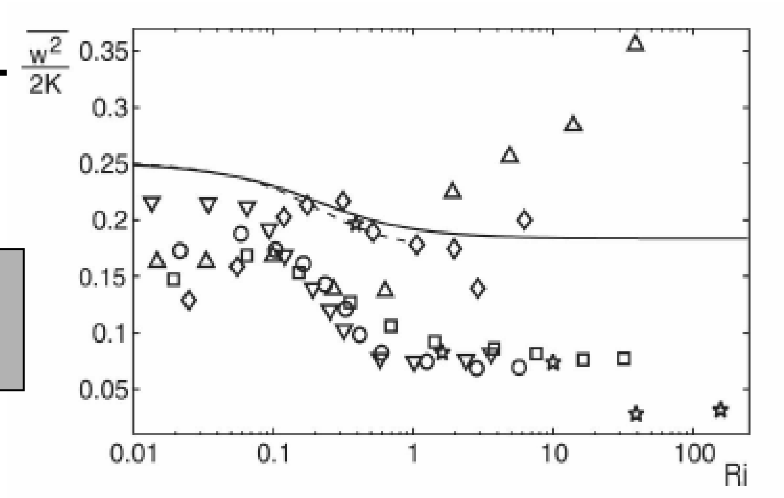


Observational data by Monti et al. (2002) and from Halley Base, Antarctica, collected by the British Antarctic Survey

Both the data and the theory point to the absence of critical Ri

Brief digression on ‘anisotropy’ (1/2)

The anisotropy of turbulent flows should not be neglected



- As shown above, anisotropy is an observed property of the flow. Full isotropy of turbulence is only obtained in the ‘free convective limit’. With increasing stability, the variance of ‘***w***’ logically diminishes with respect to those of ‘***u***’ & ‘***v***’.
- This behaviour is a key element to understand why there is no critical $\boldsymbol{R_i}$ number at very high stability (see later).

Brief digression on ‘anisotropy’ (2/2)

- And yet:
  - Some schemes, like CBR, neglect the effect of anisotropy;
  - Complex schemes that fully take it into account do not necessary lead to a solution without critical Richardson number.
- Generally speaking, in the (CBR- and p-TKE-like) framework of one turbulent prognostic quantity (TKE):
  - The stability dependency function for momentum depends only on the anisotropy handling;
  - The stability dependency function for heat merges the effect of anisotropy with a ‘substitute’ term acting instead of the (eliminated) prognostic equation for TPE.
- In QNSE (where anisotropy is at the heart of the spectral treatment), the said ‘merge’ cannot be inverted.

A few anticipations on the role of QNSE vs. p-TKE in TOUCANS (1/2)

- Something which attracted our attention rather early was that the behaviour for $\boldsymbol{R_i \Rightarrow +\infty}$ obtained by QNSE (non-zero momentum exchange, no critical $\boldsymbol{R_i}$ but an asymptotic value for the Richardson flux number $\boldsymbol{R_{if} = R_i.(K_h/K_m)}$) was common with:
  - The version of CBR (RMC01) using a similar kind of 'filter' than QNSE (*but neither with the other 'original' CBR version, nor with the generic Mellor-Yamada-type RANS model with all degrees of freedom, filtered or not*);
  - And (*more surprisingly*) the Louis scheme, in its versions since CYCORA-bis (2000), and hence with p-TKE of course!
- This led to the fruitful idea that there could exist a way to symmetrise things:
  - Using the QNSE example to make 'separation' a goal for RANS results, this having of course to be justified on independent grounds;
  - Using the RANS formalism to 'extend' the QNSE results to the unstable range of negative $\boldsymbol{R_i}$ values.

A few anticipations on the role of QNSE vs. p-TKE in TOUCANS (2/2)

- For reasons too complex to be developed here, it appeared that this symmetrisation was indeed possible, partly thanks to the p-TKE architecture linking ‘Louis’ and ‘CBR’ concepts in a kind of ‘anticipation’.
- Furthermore, unlike for QNSE, in the ‘modified RANS’ case, the separation of the momentum and heat stability dependency functions does not imply loosing track of the ‘anisotropy’ contribution within the latter.
- This issue could not have been explored with CBR, since the latter scheme ignores the anisotropy effect.
- All this helped to understand ‘a posteriori’ why we have been very lucky with p-TKE:
  - It is wrong in some core hypotheses, but
  - It nevertheless has all the correct asymptotic functional dependencies!

Generalities concerning the ‘turbulence ⇔ diffusion’ link (1/3)

- In the ‘prognostic’ framework, which we are here interested in, and forgetting for a while the phase changes’ effects, things appear simple at first sight:
  - ‘turbulence’ solves the equations for the additional prognostic variables (TKE, TPE, fluxes, …) and may deliver ‘classical’ exchange coefficients for the computation of diffusive fluxes;
  - ‘diffusion’ happens in a rather unchanged manner with respect to the ‘diagnostic’ case; it may just provide back some input to the ensuing ‘turbulent’ step.
- As a parenthesis, one should realise that this simple duality can exist because heat and moisture transport may be assumed homothetic. In the oceanic science they indeed have no clouds to consider (lucky), but the difference in diffusivity of heat and salinity forces them to use so-called two-points closure methods (unlucky). And vice-versa for us.

Generalities concerning the ‘turbulence ⇔ diffusion’ link (2/3)

- But the above idealistic vision forgets a few issues:
  - Separating diffusion from turbulence (which we must do if we want to easily treat implicitly the phase changes’ effect) means that only TKE and TPE may be handled as prognostic (fluxes cannot be computed twice);
  - Accounting for non-locality then requires some careful thinking (it must be a diagnostic computation somehow sandwiched between two prognostic ones [respectively for TKE(TPE) and for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{q_t}$ & $\boldsymbol{s_L}$]);
  - All this (as well as the above-mentioned return from the output of diffusive calculations) raises the issue of the order of the computations and of the measures taken to ensure their linear and non-linear stability.
- On top of that, phase changes intervene (at least) twice in the process: (i) in changing the resistance to buoyancy and hence the stability dependency laws & (ii) in controlling the way to return from $\boldsymbol{q_t}$ & $\boldsymbol{s_L}$ to $\boldsymbol{q_v}$, $\boldsymbol{q_l}$, $\boldsymbol{q_i}$ & $\boldsymbol{s}$, for fluxes.

Generalities concerning the ‘turbulence ⇔ diffusion’ link (3/3)

- Except for the rather trivial ‘clear’ and ‘completely covered’ cases, both treatments of the phase changes’ effects should rather be as consistent as possible (full consistency is probably impossible).
- The ‘classical’ solution to this problem (since Sommeria and Deardorff) is to link both:
  - The statistical assumptions made for obtaining a sug-grid thermodynamic adjustment between water vapour, condensates and temperature with the Reynolds-type computations of the turbulence;
  - The ‘area fraction’ obtained as a by-product of the adjustment with the ‘return’ computation for the fluxes.
- In the TOUCANS’ thinking we rather try to rely on a unique ‘bridge’, encompassed in the ‘Shallow Convective Cloudiness’ (SCC).

Precisions about the role of SCC

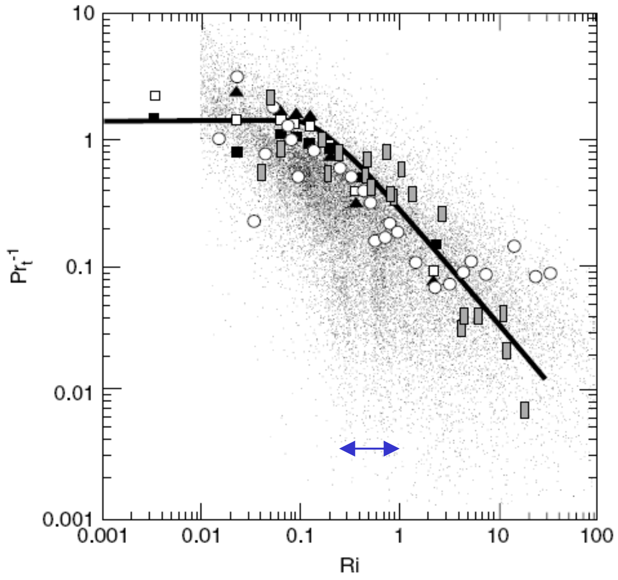
- One may postulate that the problem of the ‘classical’ solution for the diffusive transport of condensate lies in the staggering (the ‘adjustment information’ being on full-levels vs. the split of the total moisture fluxes being done on half-levels).
- The alternative solution of separate transport computations appears noise-prone (discontinuous fields).
- Having SCC (i) in order to compute moist stability dependencies in the turbulence & (ii) in order to decide the above mentioned ‘split’ seems an attractive solution.
- It also offers the chance of a complete and logical input to radiative cloudiness, even if the staggering problem is displaced there.
- If ***SCC=>0***, no problem. If ***SCC=>1***, the turbulence computation becomes ‘very stable’, diffusive transport nearly vanishes and hence the ‘full re-projection’ of total moisture diffusive transport acts on quite small fluxes, as should be.

TOUCAN's basic choices

- Solving for turbulence before computing exchange coefficients and TOM's related quantities for diffusion (p-TKE inheritance); possible a-posteriori correction of $\boldsymbol{E^{+}}$.
- One single turbulent additional prognostic variable, TKE.
- 'Filtering' use of the stationarity equation.
- Accounting for anisotropy.
- Intentional separation of the stability dependencies for momentum on the one side and for heat/moisture on the other side.
- Merging the RANS and QNSE formalisms => 'no critical Richardson number' solution / 3 parameter & 3 equation system.
- Introducing TOMs effects in a way conceptually as close as possible to a mass-flux parameterisation.
- Using the above-connected 'dry' results for moist turbulence just by redefining '***N***', the so-called Brunt-Vaisala frequency.
- Handling the 'moist' link between turbulence and diffusion consistently with the above, around the SCC concept.

The “No Ri(cr)” issue, facts

At very high stability there appears to be no limitation on the Richardson-number (but there exist a critical flux-Richardson-number R_{ifc}). There are in fact two stable regimes, one with a constant ratio of the exchange coefficients (strong mixing), one with this ratio inversely proportional to R_i (weak mixing).



The blue double arrow in the bottom part of the diagram indicates where previous theories did put the ‘critical $\boldsymbol{R_i}$’! Even if the academic question about the occurrence of infinite $\boldsymbol{R_i}$ values may be debated, it is sure that the hypotheses leading to critical $\boldsymbol{R_i}$ thresholds around [0.25-1.] must be considered as false in view of the accumulated observational evidence.

The “No Ri(cr)” issue, interpretation

- There are several ways to interpret this radical change in the ‘constraints’ which the result of any turbulence theory must obey. The most simple and convincing at the same time is probably the one given by Zilitinkevitch et al. (QJRMS, 2008).
- One should depart from the analysis of the sole budget equation for **TKE** and also look at the same budget for **TPE**. The buoyancy term of the former has an opposite equivalent in the latter (pure conversion of energy). So, apart from advective-diffusive redistribution terms, the **TTE=TKE+TPE** always has a shear source term, balanced on average by total dissipation. Hence turbulence can never disappear, QED.
- In practice it works as follows. Suppose that, for quasi-infinite stability, the buoyancy term become so strongly negative that TKE considerably decreases. The TPE increases in the same amount and fluctuations of buoyancy are strengthened. Fluid elements thus acquire stronger accelerations and some TKE is recreated (mostly horizontally). When it is becoming too big, the above-described ‘cycle’ will restart.
- This explains why the asymptotic ratio $\boldsymbol{K_m/K_h}$ must be proportional to $\boldsymbol{R_i}$. Any other power law enters in contradiction with “***0 < TPE/TTE < 1***”!

Still avoiding the additional prognostic equation for TPE

- One (rather alibistic) reason to do so is that QNSE is incompatible with it. The 'conversion' term effects are de facto incorporated into the stability dependency function for heat (remember that the 'cyclic' process previously described is surely associated with some anisotropic waves).
- More purposefully, if the prognostic TPE equation disappears, it just means that its redistribution terms are cancelled. Doing the same on the TKE side in order to get a 'filtering condition' (leading to expressions for K_m and K_h in $R_i=N^2/S^2$ rather than in N and S) restores the lost balance. The 'filtering condition' [introduced for CBR by Redelsperger et al. (BLM, 2001)] is just the expression of the conservation of TTE in a differing context. It is hence automatically compatible with 'No Ri(cr)' solutions.
- The spectacular demonstration of how one part of the stability dependency function for heat acts as a substitute for the removed prognostic TPE equation is given in the Appendix of Cheng et al. (JAS, 2002), a worth reading paper.

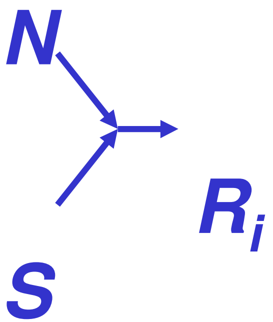
Graphical representation of the previously expressed ideas

Elimination

$$\cancel{\frac{\mathrm{D}E_{\mathrm{P}}}{\mathrm{D}t} + \frac{\partial \Phi_{\mathrm{P}}}{\partial z}} = -\beta\, F_z - \varepsilon_{\mathrm{P}}$$

Conversion term

'Filter'



$$\cancel{\frac{\mathrm{D}E_{\mathrm{K}}}{\mathrm{D}t} + \frac{\partial \Phi_{\mathrm{K}}}{\partial z}} = -\tau \cdot \overline{\mathbf{S}} + \beta\ F_z - \varepsilon_{\mathrm{K}}$$

$$\frac{\mathrm{D}E}{\mathrm{D}t} + \frac{\partial \Phi_E}{\partial z} = -\tau \cdot \overline{\mathbf{S}} - \varepsilon_E$$

Synthesis: simplifying the RANS models to a tractable set of equations for turbulence

- The next two viewgraphs show a spectacular reduction of complexity for Reynolds stress modelling equations, obtained owing to the above-mentioned steps (including the filtering role of the TKE stationarity equation).
- The advantages are multiple:
  - The stability dependency functions can be inverted => possibility to ‘parameterise’ shallow convection via a single modification of the Brunt-Vaisala frequency;
  - The inclusion of Third Order Moment (TOMs) terms can be performed at relative little computing expense;
  - The QNSE spectral theory can be well approximated within this new framework. This allows to extend its scope to the unstable range and to make it benefit from the advances mentioned in both previous bullets.

Original equations (Cheng et al.)

$$\overline{u^2} = \frac{1}{3}q^2 - \frac{\tau}{3}\left[(\lambda_1 + 3\lambda_2)\frac{\partial U}{\partial z}\overline{uw} - 2\lambda_2\frac{\partial V}{\partial z}\overline{vw} + 2\lambda_5 g\alpha\overline{w\theta}\right] \quad (15a)$$

$$\overline{v^2} = \frac{1}{3}q^2 - \frac{\tau}{3}\left[(\lambda_1 + 3\lambda_2)\frac{\partial V}{\partial z}\overline{vw} - 2\lambda_2\frac{\partial U}{\partial z}\overline{uw} + 2\lambda_5 g\alpha\overline{w\theta}\right] \quad (15b)$$

$$\overline{w^2} = \frac{1}{3}q^2 + \frac{\tau}{3}\left[(3\lambda_1 - \lambda_2)\left(\frac{\partial U}{\partial z}\overline{uw} + \frac{\partial V}{\partial z}\overline{vw}\right) + 4\lambda_5 g\alpha\overline{w\theta}\right] \quad (15c)$$

$$\overline{uv} = -(\lambda_1 + \lambda_2)\frac{\tau}{2}\left(\frac{\partial V}{\partial z}\overline{uw} + \frac{\partial U}{\partial z}\overline{vw}\right) \quad (15d)$$

$$\overline{uw} = -\frac{\tau}{2}\frac{\partial U}{\partial z}\left[\frac{1}{2}\left(\lambda_1 - \frac{4}{3}\lambda_2\right)q^2 + (\lambda_1 - \lambda_2)\overline{u^2} + (\lambda_1 + \lambda_2)\overline{w^2}\right] - (\lambda_1 - \lambda_2)\frac{\tau}{2}\frac{\partial V}{\partial z}\overline{uv} + \lambda_5\tau g\alpha\overline{u\theta} \quad (15e)$$

$$\overline{vw} = -\frac{\tau}{2}\frac{\partial V}{\partial z}\left[\frac{1}{2}\left(\lambda_1 - \frac{4}{3}\lambda_2\right)q^2 + (\lambda_1 - \lambda_2)\overline{v^2} + (\lambda_1 + \lambda_2)\overline{w^2}\right] - (\lambda_1 - \lambda_2)\frac{\tau}{2}\frac{\partial U}{\partial z}\overline{uv} + \lambda_5\tau g\alpha\overline{v\theta} \quad (15f)$$

$$\overline{u\theta} = -\lambda_3^{-1}\tau\left[\frac{\partial\Theta}{\partial z}\overline{uw} + \frac{1}{2}(\lambda_6 + \lambda_7)\frac{\partial U}{\partial z}\overline{w\theta}\right] \quad (15g)$$

$$\overline{v\theta} = -\lambda_3^{-1}\tau\left[\frac{\partial\Theta}{\partial z}\overline{vw} + \frac{1}{2}(\lambda_6 + \lambda_7)\frac{\partial V}{\partial z}\overline{w\theta}\right] \quad (15h)$$

$$\overline{w\theta} = -\lambda_3^{-1}\tau\left[\frac{\partial\Theta}{\partial z}\overline{w^2} + \frac{1}{2}(\lambda_6 - \lambda_7)\left(\frac{\partial U}{\partial z}\overline{u\theta} + \frac{\partial V}{\partial z}\overline{v\theta}\right)\right] \times \left[1 + \lambda_3^{-1}\lambda_8 g\alpha\tau^2\frac{\partial\Theta}{\partial z}\right]^{-1}. \quad (15i)$$

Equations (15a)–(15i) can be solved using symbolic algebra. The results are

$$(\overline{uw}, \overline{vw}) = -K_M\left(\frac{\partial U}{\partial z}, \frac{\partial V}{\partial z}\right) \quad (16a)$$

$$\overline{w\theta} = -K_H\frac{\partial\Theta}{\partial z} \quad (16b)$$

$$K_M = e\tau S_M, \qquad K_H = e\tau S_H \quad (16c)$$

$$S_M = \frac{1}{D}(s_0 + s_1 G_H + s_2 G_M) \quad (17a)$$

$$S_H = \frac{1}{D}(s_4 + s_5 G_H + s_6 G_M), \quad (17b)$$

where G_H and G_M are defined as

$$G_H = (\tau N)^2, \qquad G_M = (\tau S)^2 \quad (18a)$$

$$N^2 = g\alpha\frac{\partial\Theta}{\partial z}, \qquad S^2 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \quad \text{and} \quad (18b)$$

$$D = 1 + d_1 G_H + d_2 G_M + d_3 G_H^2 + d_4 G_H G_M + d_5 G_M^2 \quad (18c)$$

$$d_1 = \lambda_3^{-1}\left(\frac{7}{3}\lambda_6 + \lambda_8\right)$$

$$d_2 = \left(\lambda_1^2 - \frac{1}{3}\lambda_2^2\right) - \frac{1}{4}\lambda_3^{-2}(\lambda_6^2 - \lambda_7^2),$$

$$d_3 = \frac{1}{3}\lambda_5\lambda_3^{-2}(4\lambda_6 + 3\lambda_8)$$

$$d_4 = \frac{1}{3}\lambda_5\lambda_3^{-2}[\lambda_2\lambda_6 - 3\lambda_1\lambda_7 - \lambda_3(\lambda_1^2 - \lambda_2^2)] + \lambda_3^{-1}\lambda_8\left(\lambda_1^2 - \frac{1}{3}\lambda_2^2\right)$$

$$d_5 = -\frac{1}{4}\lambda_3^{-2}\left(\lambda_1^2 - \frac{1}{3}\lambda_2^2\right)(\lambda_6^2 - \lambda_7^2), \qquad s_0 = \frac{1}{2}\lambda_1$$

$$s_1 = -\frac{1}{3}\lambda_5\lambda_3^{-2}(\lambda_6 + \lambda_7) + \frac{2}{3}\lambda_5\lambda_3^{-1}\left(\lambda_1 - \frac{1}{3}\lambda_2 - \lambda_3\right) + \frac{1}{2}\lambda_1\lambda_3^{-1}\lambda_8$$

$$s_2 = -\frac{1}{8}\lambda_1\lambda_3^{-2}(\lambda_6^2 - \lambda_7^2), \qquad s_4 = \frac{2}{3}\lambda_3^{-1},$$

$$s_5 = \frac{2}{3}\lambda_5\lambda_3^{-2}$$

$$s_6 = \frac{2}{3}\lambda_3^{-1}\left(\lambda_1^2 - \frac{1}{3}\lambda_2^2\right) - \frac{1}{2}\lambda_1\lambda_3^{-1}\left(\lambda_1 - \frac{1}{3}\lambda_2\right) + \frac{1}{4}\lambda_1\lambda_3^{-2}(\lambda_6 - \lambda_7). \quad (18d)$$

Resulting set of phenomenological equations

$$R_{if} = C_3 R_i \frac{\phi_3(R_i)}{\chi_3(R_i)}$$

$$\chi_3(R_i) = \frac{1 - R_{if} / R}{1 - R_{if}}$$

$$\phi_3(R_i) = \frac{1 - R_{if} / R_{ifc}}{1 - R_{if}}$$

C_3 : inverse Prandtl number at neutrality

R : parameter characterising the flow's anisotropy

R_{ifc} : critical flux-Richardson number (***R_{if}*** at $+\infty$)

Plus the 'developed' prognostic TKE equation (for 'E')

$$\frac{\partial E}{\partial t} = A_{dv}(E) + \frac{1}{\rho}\frac{\partial}{\partial z}\frac{\rho K_m}{\sqrt{C_K C_\varepsilon}}\frac{\partial E}{\partial z} + K_m\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] - \frac{g}{\theta}K_h\frac{\partial \theta}{\partial z} - \frac{C_\varepsilon E^{3/2}}{L}$$

$$K_m = C_K L \chi_3(R_i)\sqrt{E} \qquad K_h = C_3 C_K L \phi_3(R_i)\sqrt{E}$$

L *length scale* $\qquad C_K$ & C_ε *tuning parameters*

Non-locality, generalities

- It has long been recognised (via comparisons with Reynolds stress models using a higher order closure) that the pure down-gradient terms cannot fully reproduce the complex solutions observed in the PBL.
- Historically, the incorporation of non-locality has progressively gone from the fully heuristic to the NWP-compatible:
  - Modification of the θ gradients by a constant ‘counter gradient correction’ (in order to get upward fluxes even in slightly stable cases, like below inversions);
  - Addition of empirical but time- and space-variable corrections;
  - Higher order closure models recast in terms of standard vertical profiles of corrections;
  - The recent ‘compact’ work of Canuto et al., which will be the basis for the solution proposed in TOUCANS.

Also in the domain of higher order moments, one is indeed looking for some reduction of complexity

together with Eqs. (12g) and (12h) for a total of seven dynamic equations like in the shear driven case. The third-order moments are taken to be the steady state solutions of Eqs. (2a)–(2c) of Appendix B, which, using the forth-order moments discussed in Appendix C, become (Cheng et al., 2005):

$$\overline{w^3} = -A_1 \frac{\partial}{\partial z}\overline{w^2} - A_2 \frac{\partial}{\partial z}\overline{w\theta} - A_3 \frac{\partial}{\partial z}\overline{\theta^2} \tag{14a}$$

$$\overline{w^2\theta} = -A_4 \frac{\partial}{\partial z}\overline{w^2} - A_5 \frac{\partial}{\partial z}\overline{w\theta} - A_6 \frac{\partial}{\partial z}\overline{\theta^2} \tag{14b}$$

$$\overline{w\theta^2} = -A_7 \frac{\partial}{\partial z}\overline{w\theta} - A_8 \frac{\partial}{\partial z}\overline{\theta^2} \tag{14c}$$

All the third-order moments exhibit a linear combination of the z-derivatives of the second-order moments, as first discussed in Canuto et al. (1994). In (14a–c), the "diffusivities" A's are given by ($\lambda = (1 - c_{11})g\alpha_T$):

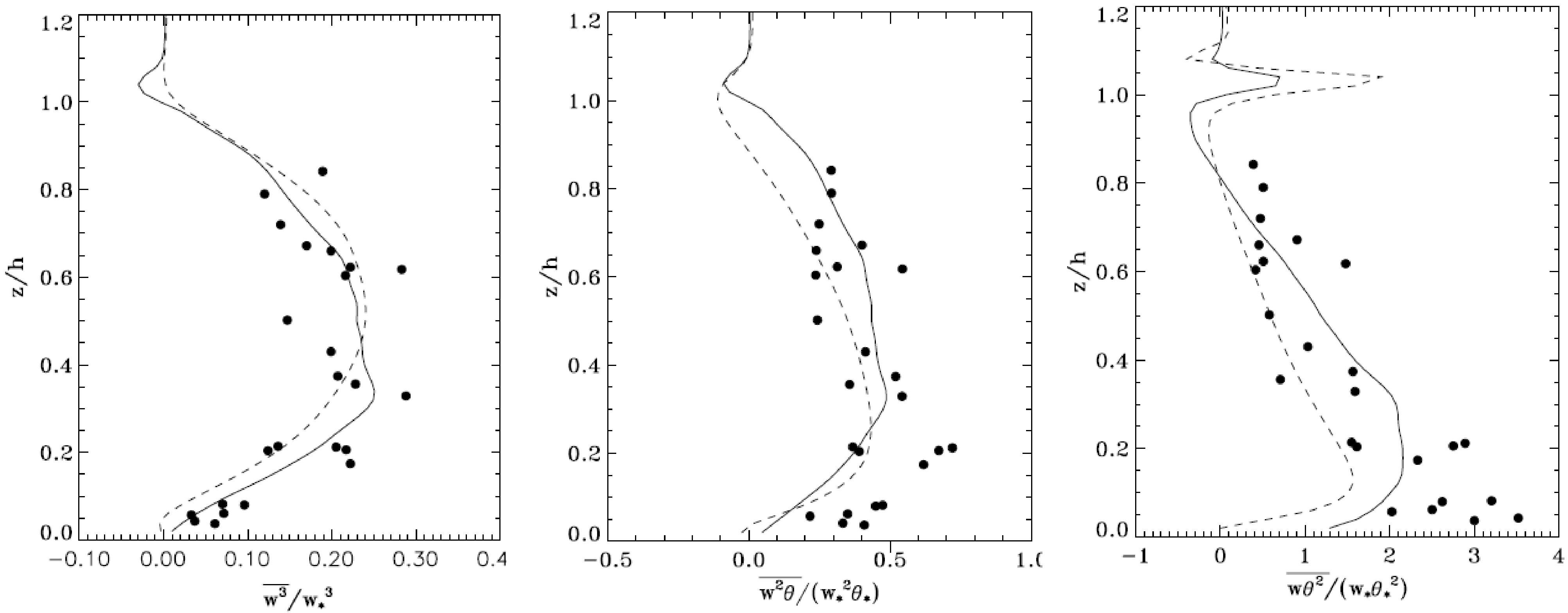
$$\begin{aligned}
A_1 &= \left(a_1\overline{w^2} + a_2\lambda\tau\overline{w\theta}\right)\tau, \quad A_2 = \left(a_3\overline{w^2} + a_4\lambda\tau\overline{w\theta}\right)\lambda\tau^2 \\
A_3 &= \left(a_5\overline{w^2} + a_6\lambda\tau\overline{w\theta}\right)\lambda^2\tau^3, \quad A_4 = a_7\tau\overline{w\theta} \\
A_5 &= \left(a_8\overline{w^2} + a_9\lambda\tau\overline{w\theta}\right)\tau, \quad A_6 = \left(a_{10}\overline{w^2} + a_{11}\lambda\tau\overline{w\theta}\right)\lambda\tau^2 \\
A_7 &= a_{12}\tau\overline{w\theta}, \quad A_8 = \left(a_{13}\overline{w^2} + a_{14}\lambda\tau\overline{w\theta}\right)\tau
\end{aligned} \tag{14d}$$

The coefficients a_k's in (14d) are listed in Table 1.

Even though Eqs. (14a)–(14d) are relatively simple and have been successfully tested against LES data (Cheng et al., 2005), more recently we have succeeded in reducing them even further without deteriorating the comparison with LES data. In fact, we have found the following simplified version of (14a)–(14d):

$$\overline{w^3} = -0.06g\alpha\tau^2\overline{w^2}\frac{\partial\overline{w\theta}}{\partial z}, \quad \overline{w^2\theta} = -0.3\tau\overline{w^2}\frac{\partial\overline{w\theta}}{\partial z}, \quad \overline{w\theta^2} = -\tau\overline{w\theta}\frac{\partial\overline{w\theta}}{\partial z} \tag{14e}$$

Proof with fit to aircraft data (dots) and to LES computations (dashed lines)



Canuto et al. (Ocean Modelling, 2007)

The quality with three terms is nearly as good as with fourteen ones (and this number 14 was already obtained thanks to a simplification in the jump from 4th order to 3rd order moments)!

Resulting set of 'corrective equations'

$$J_h = \rho K_h \frac{\partial\theta}{\partial z} + \rho K_h (Q\phi_Q)^{-1}\overline{\mathrm{T}_*^{-1}\frac{\partial J_h}{\partial p}} \quad with \quad \mathrm{T}_*^{-1} = \left(\mathrm{T}_*^{-1}\right)_0 - \left(\mathrm{T}_*^{-1}\right)_\theta \frac{\partial}{\partial p}\left[\tau (J_h)_0\right]$$

$$(\mathrm{T}_*^{-1})_0 = \frac{6C_0^g}{5c_\varepsilon^{\ 2}}\frac{g^2}{e}\left[\frac{\partial}{\partial p}\left(\rho\frac{L^2(Q\varphi_Q)}{\tau}\right) - \frac{6}{125 c_\varepsilon c_K}\overline{\left(\frac{R_i}{\chi_3(1-R_{if})\tau}\right)}\frac{\partial}{\partial p}\left(\rho L^2(Q\phi_Q)\right)\right]$$

$$(\mathrm{T}_*^{-1})_\theta = \frac{C_1^g c_\varepsilon}{4c_\theta}\left[\frac{g}{\theta}\overline{\tau}\right]\frac{g^2}{e}$$

$$\frac{\partial\theta}{\partial t} = -g\frac{\partial}{\partial p}\left[\rho K_h \frac{\partial\theta}{\partial z}\right] + \frac{\partial}{\partial p}\left[\rho K_h (Q\phi_Q)^{-1}\overline{\mathrm{T}_*^{-1}\frac{\partial\theta}{\partial t}}\right]$$

$$(S_{cg})_0 = \frac{\partial}{\partial p}\left[\rho K_h (Q\phi_Q)^{-1}\overline{\mathrm{T}_*^{-1}(\theta^* - \theta^-)}\right] \quad \left(when\ solving\ \frac{\partial\theta}{\partial t} = -g\frac{\partial}{\partial p}\left[\rho K_h\frac{\partial\theta}{\partial z}\right]\ to\ get\ (J_h)_0\right)$$

$$\delta\theta^+ = -g\,\delta t\,\frac{\partial}{\partial p}\left[\rho K_h\frac{\partial(\delta\theta^+)}{\partial z}\right] + \frac{\partial}{\partial p}\left[\rho K_h (Q\phi_Q)^{-1}\overline{\mathrm{T}_*^{-1}\delta\theta^+}\right] + (S_{cg})_0$$

Last remarks on the TOMs' terms handling

- A similar set of corrective equations can be derived for moisture (but not for momentum).
- Looking at the core equation

$$\frac{\partial \theta}{\partial t} = -g\frac{\partial}{\partial p}\left[\rho K_h \frac{\partial \theta}{\partial z}\right] + \frac{\partial}{\partial p}\left[\rho K_h (Q\phi_Q)^{-1}\overline{\mathrm{T}_*^{-1}\frac{\partial \theta}{\partial t}}\right]$$

one sees that the second term on the right hand side is of the mass-flux type. Simply, in the spirit of the 'counter gradient' denomination, the sign of the mass-flux cannot be anticipated.

- This creates a discretisation problem, but not a physical one, on the contrary: the 'convergence' of the corrections to fluxes due to TOM's terms is part of the description of what happens near inversions.

Outlook

- Although not all details were treated, it is hoped that you got a good flavour of the TOUCANS' spirit and architecture.
- The presentation of those as a logical construction is a pure 'hindcast' setting. In reality:
  - We first wanted to emulate f-TKE with the p-TKE solver;
  - Then came the inclusion of the QNSE problematic, with the aim of symmetrising the situation with RANS computations;
  - Losing a bit of faith in '3MT for everything', we got interested in the TOMs' inclusion as an alternative, also to 'mixed' schemes;
  - Finally the moist turbulent problematic forced us to reconsider the role of SCC and the vertical staggering problem in the 'turbulence ⇔ diffusion' link.
- Ivan will give you some overview of the 'new turbulence' ensemble, plus some specific details, before I come back, also in some details, to a few aspects of the above.